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On Fuzzy Proximity Spaces

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1. INTRODUCTION

The concept of a fuzzy set was introduced by Zadeh in [13] and was used afterwards by many other authors in various branches of mathematics. In [1] Chang defined fuzzy topological spaces. Several others continued the investigation of such spaces. Among the papers dealing with this subject are [3, 4, 6–9, 11, 12].

In [5] the author defined the fuzzy proximity spaces and studied some of their properties. In this paper we continue the study of fuzzy proximity spaces and prove some results analogous to those that hold for ordinary proximity spaces (see [10]).

2. PRELIMINARIES

Let X be a nonempty set and I the unit interval. A fuzzy set in X is an element of the set I^X of all functions from X to I. If f is a function from X to Y and $\mu \in I^Y$, then $f^{-1}(\mu)$ is the element of I^X which is defined by $f^{-1}(\mu)(x) = \mu(f(x))$. Also, for $\sigma \in I^X$, $f(\sigma)$ is the member of I^Y defined by

$$\begin{aligned} f(\sigma)(y) &= \sup_{x \in f^{-1}[y]} \sigma(x) \qquad \text{if} \quad f^{-1}[y] \text{ is not empty} \\ &= 0 \qquad\qquad\qquad\quad \text{otherwise.} \end{aligned}$$

A fuzzy topology on X is a subset α of I^X such that

(i) $0, 1 \in \alpha$.

(ii) If $\mu, \rho \in \alpha$, then $\mu \wedge \rho \in \alpha$.

(iii) If $\mu_i \in \alpha$ for each $i \in A$, then $\sup_{i \in A} \mu_i \in \alpha$.

A map $\mu \mapsto \bar{\mu}$, from I^X into I^X, is said to be a closure operator if for all $\mu, \rho \in I^X$ we have

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(1) $\mu \leqslant \bar{\mu}$.

(2) $\bar{\bar{\mu}} = \bar{\mu}$.

(3) $\overline{\mu \vee \rho} = \bar{\mu} \vee \bar{\rho}$.

(4) $\bar{0} = 0$.

Given a closure operator on I^X, the collection

$$\{\mu \in I^X: \overline{1 - \mu} = 1 - \mu\}$$

is a fuzzy topology on X.

A binary relation δ on I^X is called a fuzzy proximity on X if δ satisfies the following axioms:

(FP1) $\mu \,\delta\, \rho$ implies $\rho \,\delta\, \mu$.

(FP2) $(\mu \vee \rho) \,\delta\, \sigma$ iff $\mu \,\delta\, \sigma$ or $\rho \,\delta\, \sigma$.

(FP3) $\mu \,\delta\, \rho$ implies $\mu \neq 0$ and $\rho \neq 0$.

(FP4) $\mu \,\bar{\delta}\, \rho$ implies that there exists a $\gamma \in I^X$ such that $\mu \,\bar{\delta}\, \gamma$ and $(1 - \gamma) \,\bar{\delta}\, \rho$.

(FP5) $\mu \wedge \rho \neq 0$ implies $\mu \,\delta\, \rho$.

If δ is a fuzzy proximity on X, the pair (X, δ) is called a fuzzy proximity space. It is shown in [5] that if (X, δ) is a fuzzy proximity space, then

(a) The map $\mu \mapsto \bar{\mu} = 1 - \sup\{\rho \in I^X: \rho \,\bar{\delta}\, \mu\}$ is a closure operator on I^X.

(b) If $\mu \,\delta\, \rho$ and $\mu_1 \geqslant \mu$, $\rho_1 \geqslant \rho$, then $\mu_1 \,\delta\, \rho_1$.

(c) $\mu \,\delta\, \rho$ iff $\bar{\mu} \,\delta\, \rho$.

A map f, from a fuzzy proximity space (X, δ_1) to a fuzzy proximity space (Y, δ_2), is called a proximity map if $\mu \,\delta_1\, \rho$ implies $f(\mu) \,\delta_2\, f(\rho)$. Equivalently, f is a proximity map if $\mu \,\bar{\delta}_2\, \rho$ implies $f^{-1}(\mu) \,\bar{\delta}_1\, f^{-1}(\rho)$.

2.1. Proposition. *Let (X, δ) be a fuzzy proximity space and $\mu \in I^X$. If $A = \{x \in X: \bar{\mu}(x) \neq 0\}$, then $\bar{\mu}$ coincides with the characteristic function χ_A of A.*

Proof. Let $x \in A$. If $\rho \in I^X$ is such that $\mu \,\bar{\delta}\, \rho$, then $\bar{\mu} \,\bar{\delta}\, \rho$ and hence $\bar{\mu} \wedge \rho = 0$. Hence $\rho(x) = 0$ (since $\bar{\mu}(x) \neq 0$). Thus $\sup\{\rho(x): \rho \,\bar{\delta}\, \mu\} = 0$ and therefore $\bar{\mu}(x) = 1$.

3. An Alternate Description of a fuzzy Proximity

3.1. Definition. Let (X, δ) be a fuzzy proximity space. For $\mu, \rho \in I^X$ we say that ρ is a δ-*neighborhood* of μ (in symbols $\mu \ll \rho$) if $\mu \,\bar{\delta}\, (1 - \rho)$.

3.2. PROPOSITION. *Let (X, δ) be a fuzzy proximity space and $\mu, \rho \in I^X$. Then:*

(1) *$\mu \ll \rho$ implies $\bar{\mu} \ll \rho$.*

(2) *$\mu \ll \rho$ implies that there exists an element μ_1 of the fuzzy topology $\tau(\delta)$ induced by δ on X such that $\mu \leqslant \mu_1 \leqslant \rho$.*

(3) *If $\mu \bar{\delta} \rho$, there are $\gamma_1, \gamma_2 \in I^X$ with $\mu \ll \gamma_1$, $\rho \ll \gamma_2$ and $\gamma_1 \bar{\delta} \gamma_2$.*

Proof. (1) This follows directly from the definition of $\ll$ since, for $\gamma \in I^X$, $\bar{\mu} \delta \gamma$ implies $\mu \delta \gamma$.

(2) Suppose that $\mu \ll \rho$. Then $\mu \bar{\delta} (1 - \rho)$ and hence $\overline{1 - \rho} \leqslant 1 - \mu$. Thus

$$1 - \rho \leqslant \overline{1 - \rho} \leqslant 1 - \mu.$$

If $\mu_1 = 1 - \overline{1 - \rho}$, then $\mu_1 \in \tau(\delta)$ and

$$\mu \leqslant \mu_1 \leqslant \rho.$$

(3) Let $\mu \bar{\delta} \rho$. By (FP4), there exists γ_2 such that $\mu \bar{\delta} \gamma_2$ and $(1 - \gamma_2) \bar{\delta} \rho$. Similarly from the $\mu \bar{\delta} \gamma_2$ it follows that there exists γ_1 such that $\mu \bar{\delta} (1 - \gamma_1)$ and $\gamma_1 \bar{\delta} \gamma_2$. Now $\mu \ll \gamma_1$, $\rho \ll \gamma_2$, and $\gamma_1 \bar{\delta} \gamma_2$.

3.3. THEOREM. *If (X, δ) is a fuzzy proximity space, then the binary relation $\ll$ on I^X has the following properties:*

(1) *$1 \ll 1$.*

(2) *$\mu \ll \rho$ implies $\mu \wedge (1 - \rho) = 0$.*

(3) *If $\mu_1 \leqslant \mu \ll \rho \leqslant \rho_1$, then $\mu_1 \ll \rho_1$.*

(4) *$\mu \ll \mu_1 \wedge \mu_2 \wedge \cdots \wedge \mu_n$ iff $\mu \ll \mu_i$ for $i = 1, \ldots, n$.*

(5) *$\mu \ll \rho$ implies $(1 - \rho) \ll (1 - \mu)$.*

(6) *If $\mu \ll \rho$, then there exists a γ such that $\mu \ll \gamma \ll \rho$.*

Conversely, if $\ll$ is a binary relation on I^X satisfying (1)–(6), then the binary relation δ on I^X, defined by $\mu \bar{\delta} \rho$ iff $\mu \ll (1 - \rho)$, is a fuzzy proximity on X. With respect to this fuzzy proximity, ρ is a δ-neighborhood of μ iff $\mu \ll \rho$.

Proof. Parts (1) and (2) follow from the (FP3) and (FP5), respectively.

Parts (3) and (5) are easy consequences of the definitions.

Since $1 - (\mu_1 \wedge \mu_2 \wedge \cdots \wedge \mu_n) = (1 - \mu_1) \vee (1 - \mu_2) \vee \cdots \vee (1 - \mu_n)$ and since $\mu \delta (1 - \mu_1) \vee \cdots \vee (1 - \mu_n)$ iff $\mu \delta (1 - \mu_i)$ for some i, (4) follows directly from the definition of $\ll$.

Finally for the (6), suppose that $\mu \ll \rho$. Then $\mu \bar{\delta} (1 - \rho)$. By (FP4), there exists a $\gamma \in I^X$ with $\mu \bar{\delta} (1 - \gamma)$ and $\gamma \bar{\delta} (1 - \rho)$. Clearly $\mu \ll \gamma \ll \rho$. Conversely, let $\ll$ be a binary relation on I^X which satisfies (1)–(6). We define δ on I^X by

$$\mu \bar{\delta} \rho \quad \text{iff} \quad \mu \ll (1 - \rho). \tag{*}$$

We will show that δ satisfies (FP1)–(FP5).

(FP1) This follows from (5).

(FP2) Suppose that $\mu \bar{\delta} \sigma$ and $\rho \bar{\delta} \sigma$. Then $\sigma \bar{\delta} \mu$, $\sigma \bar{\delta} \rho$ and so $\sigma \ll (1 - \mu)$ and $\sigma \ll (1 - \rho)$. By (4), $\sigma \ll (1 - \mu) \wedge (1 - \rho)$. Thus $\sigma \ll 1 - (\mu \vee \rho)$ and so $\sigma \bar{\delta} (\mu \vee \rho)$.

(FP3) Since $\mu \leqslant 1 \ll 1$, it follows that $\mu \ll 1$ and hence $\mu \bar{\delta} 0$.

(FP4) Suppose that $\mu \bar{\delta} \rho$. Then $\mu \ll (1 - \rho)$. By (6), there exists a γ such that $\mu \ll \gamma \ll (1 - \rho)$. Clearly $\mu \bar{\delta} (1 - \gamma)$ and $\gamma \bar{\delta} \rho$.

(FP5) If $\mu \bar{\delta} \rho$, then $\mu \ll (1 - \rho)$ and hence $\mu \wedge \rho = 0$ by (2).

Finally, it is clear that ρ is a δ-neighborhood of μ iff $\mu \ll \rho$.

3.4. Theorem. *Let (X, δ) be a fuzzy proximity space and $\mu \in I^X$. Then*

$$\bar{\mu} = \inf\{\rho \in I^X : \mu \ll \rho\}.$$

Proof. Clearly there are $\rho \in I^X$ with $\mu \ll \rho$. Let $\gamma = \inf\{\rho : \mu \ll \rho\}$. If $\mu \ll \rho$, then $\bar{\mu} \ll \rho$ and hence $\bar{\mu} \leqslant \rho$ (Proposition 3.2). Thus $\bar{\mu} \leqslant \gamma$. On the other hand, let $x \in X$ and $\epsilon > 0$. There exists a $\rho \in I^X$ with $\mu \bar{\delta} \rho$ and $1 - \rho(x) < \bar{\mu}(x) + \epsilon$. If $\rho_1 = 1 - \rho$, then $\mu \ll \rho_1$ and thus

$$\gamma(x) \leqslant \rho_1(x) < \bar{\mu}(x) + \epsilon.$$

This completes the proof.

4. Fuzzy Filters

4.1. Definition. Let β be a nonempty subset of I^X. Then, β is called a *base* for a fuzzy *filter* on X if the following two conditions are satisfied:

(1) $0 \notin \beta$.

(2) If $\mu_1, \mu_2 \in \beta$, then there exists $\mu_3 \in \beta$ with $\mu_3 \leqslant \mu_1 \wedge \mu_2$. If β also has the property

(3) $\mu \geqslant \rho \in \beta$ implies $\mu \in \beta$,

then β is called a *fuzzy filter* on X.

A maximal, with respect to set inclusion, fuzzy filter on X is called a *fuzzy ultrafilter* on X.

Clearly, if β is a base for a fuzzy filter on X, the collection

$$\varphi(\beta) = \{\mu \in I^X : \exists\, \rho \in \beta \text{ with } \rho \leqslant \mu\}$$

is a fuzzy filter on X. If $\varphi(\beta)$ is a fuzzy ultrafilter on X, then β is called a maximal fuzzy base (or a fuzzy ultrafilter base) on X.

Using Zorn's lemma, we show easily that every fuzzy filter on X is contained in a fuzzy ultrafilter.

One can easily show the following.

4.2. LEMMA. *Let σ be a nonempty subset of I^X. The following are equivalent:*

(a) *If $\mu_1, \mu_2, \ldots, \mu_n$ are arbitrary elements of σ, then $\mu_1 \wedge \mu_2 \wedge \cdots \wedge \cdots \wedge \mu_n \neq 0$.*

(b) *There is a fuzzy filter φ on X containing σ.*

If $\sigma \subset I^X$ has the property (a) of the preceding lemma, then the set

$$\{\mu \in I^X : \exists\, \mu_1, \ldots, \mu_n \in \sigma \text{ with } \mu_1 \wedge \cdots \wedge \mu_n \leqslant \mu\}$$

is a fuzzy filter on X containing σ and σ is called a subbase for this filter.

4.3. PROPOSITION. *Let φ be a fuzzy filter on X. Then*

(1) *φ is a fuzzy ultrafilter on X iff every $\mu \in I^X$ with $\mu \wedge \rho \neq 0$ for all $\rho \in \varphi$ belongs to φ.*

(2) *If φ is a fuzzy ultrafilter on X and $\mu_1 \vee \mu_2 \in \varphi$, then either $\mu_1 \in \varphi$ or $\mu_2 \in \varphi$.*

(3) *If φ is a fuzzy ultrafilter on X, then for each $\mu \in I^X$ either $\mu \in \varphi$ or $1 - \mu \in \varphi$.*

Proof. (1) Suppose that φ is a fuzzy ultrafilter and let $\mu \in I^X$ be such that $\mu \wedge \rho \neq 0$ for each $\rho \in \varphi$. Let $\sigma = \varphi \cup \{\mu\}$. Then σ has the property (a) of Lemma 4.2. Hence, there exists a fuzzy filter φ_1 on X with $\sigma \subset \varphi_1$. By the maximality of φ, we have $\varphi = \varphi_1$ and thus $\mu \in \varphi$.

Conversely, suppose that φ contains every $\mu \in I^X$ such that $\mu \wedge \rho \neq 0$ for each $\rho \in \varphi$. Let φ_1 be another fuzzy filter on X which contains φ and let $\mu \in \varphi_1$. If $\rho \in \varphi$, then $\mu, \rho \in \varphi_1$ and hence $\mu \wedge \rho \neq 0$. This, by hypothesis, implies that $\mu \in \varphi$.

(2) Suppose that $\mu_1, \mu_2 \notin \varphi$ and set $\mu = \mu_1 \vee \mu_2$. By (1), there are $\rho_1, \rho_2 \in \varphi$ with $\mu_1 \wedge \rho_1 = \mu_2 \wedge \rho_2 = 0$. If $\rho = \rho_1 \wedge \rho_2$, then $\rho \in \varphi$ and $\rho \wedge \mu = 0$, which implies that $\mu \notin \varphi$.

(3) Suppose, by way of contradiction, that neither μ nor $1 - \mu$ belongs to φ. By (1) and (2), there exists a $\rho \in \varphi$ with $\rho \wedge [\mu \vee (1 - \mu)] = 0$. Since $\rho \neq 0$, there is an $x \in X$ with $\rho(x) \neq 0$. For this x we also have $\mu(x) \vee (1 - \mu(x)) \neq 0$. This contradicts the $\rho \wedge [\mu \vee (1 - \mu)] = 0$.

4.4. PROPOSITION. *Let $f: X \to Y$ be a map. If $\beta \subset I^X$ is a base for a fuzzy ultrafilter on X, then*

$$f(\beta) = \{f(\mu): \mu \in \beta\}$$

is a base for a fuzzy ultrafilter on Y.

Proof. We first show that $f(\beta)$ is a base for a fuzzy filter on Y. Clearly $f(\beta)$ is not empty and $0 \notin f(\beta)$. Let $\mu_1, \mu_2 \in \beta$. There exists a $\mu_3 \in \beta$ with $\mu_3 \leqslant \mu_1 \wedge \mu_2$. Clearly $f(\mu_3) \leqslant f(\mu_1) \wedge f(\mu_2)$. This proves that $f(\beta)$ is a base for a fuzzy filter on Y. Let φ be the fuzzy filter on Y generated by $f(\beta)$ and let φ_1 be another fuzzy filter on Y which contains φ. We will show that $\varphi = \varphi_1$. Let $\rho \in \varphi_1$ and let φ_0 be the fuzzy ultrafilter on X generated by β and let $\mu_1 \in \varphi_0$. There exists $\mu_2 \in \beta$ with $\mu_2 \leqslant \mu_1$. Since $f(\mu_2)$ and ρ belong both to the fuzzy filter φ_1, we have $f(\mu_2) \wedge \rho \neq 0$ and thus there exists an $\alpha > 0$ and $y \in Y$ with $\rho(y) > \alpha$, $f(\mu_2)(y) > \alpha$. By the definition of $f(\mu_2)$, there exists an $x \in X$ with $f(x) = y$ and $\mu_2(x) > \alpha$. Now

$$(f^{-1}(\rho) \wedge \mu_1)(x) \geqslant (f^{-1}(\rho) \wedge \mu_2)(x) = \min\{\mu_2(x), \rho(f(x))\} > \alpha.$$

This shows that $f^{-1}(\rho) \wedge \mu_1 \neq 0$ for each $\mu_1 \in \varphi_0$ and hence $f^{-1}(\rho) \in \varphi_0$ by Proposition 4.3. Hence there exists $\mu \in \beta$ with $\mu \leqslant f^{-1}(\rho)$. Now

$$\rho \geqslant f(f^{-1}(\rho)) \geqslant f(\mu)$$

and hence $\rho \in \varphi$. This shows that $\varphi_1 = \varphi$ and the proof is complete.

4.5. PROPOSITION. *Let φ be a fuzzy ultrafilter on a set X, $Y \subset X$ and suppose that there exists a $\mu_0 \in \varphi$ which vanishes on $X - Y$. For each $\mu \in I^X$ we set $\tilde{\mu} = \mu \mid Y$. Then*

$$\tilde{\varphi} = \{\tilde{\mu}: \mu \in \varphi\}$$

is a fuzzy ultrafilter on Y.

Proof. If $\mu \in I^X$ is such that $\tilde{\mu} = 0$, then $\mu \wedge \mu_0 = 0$ and hence $\mu \notin \varphi$. Thus $0 \notin \tilde{\varphi}$. Also, if $\mu_1, \mu_2 \in \varphi$, then $\mu_1 \wedge \mu_2 \in \varphi$ and $(\mu_1 \wedge \mu_2)^{\sim} = \tilde{\mu}_1 \wedge \tilde{\mu}_2$. Hence $\tilde{\mu}_1 \wedge \tilde{\mu}_2 \in \tilde{\varphi}$. Suppose next that $\rho \in I^Y$ is such that $\rho \geqslant \tilde{\mu}$ for some $\mu \in \varphi$. We define $\rho_1 \in I^X$ by

$$\begin{aligned} \rho_1(x) &= \rho(x) &\quad \text{if} \quad & x \in Y \\ &= 1 &\quad \text{if} \quad & x \notin Y. \end{aligned}$$

Then $\rho_1 \geqslant \mu$ and hence $\rho_1 \in \varphi$. Thus $\rho = \tilde{\rho}_1 \in \tilde{\varphi}$. This proves that $\tilde{\varphi}$ is a fuzzy filter on Y.

Finally, let $\rho \in I^Y$ with $\rho \wedge \tilde{\mu} \neq 0$ for each $\mu \in \varphi$. Define $\rho_1: X \to I$ by $\rho_1(x) = \rho(x)$ if $x \in Y$ and $\rho_1(x) = 1$ if $x \notin Y$. Then $\rho_1 \wedge \mu \neq 0$ for each $\mu \in \varphi$ and hence $\rho_1 \in \varphi$ since φ is a fuzzy ultrafilter. Thus $\rho = \tilde{\rho}_1 \in \tilde{\varphi}$. Thus, by Proposition 4.3, completes the proof.

5. Clusters

5.1. Definition. Let δ be a fuzzy proximity on a set X. A subset σ of I^X is called a δ-cluster if the following conditions are satisfied:

(C1) If $\mu, \rho \in \sigma$, then $\mu \, \delta \, \rho$.

(C2) If $\mu \, \delta \, \rho$ for each $\rho \in \sigma$, then $\mu \in \sigma$.

(C3) $\mu \vee \rho \in \sigma$ implies that either $\mu \in \sigma$ or $\rho \in \sigma$.

5.2. *Note.* Let (X, δ) be a fuzzy proximity space and $x \in X$. Then the set

$$\sigma_x = \{\mu \in I^X : \bar{\mu}(x) = 1\}$$

is a δ-cluster.

In fact, let $\mu, \rho \in \sigma_x$. Then $\bar{\mu} \wedge \bar{\rho} \neq 0$ and thus $\bar{\mu} \, \delta \, \bar{\rho}$, which implies that $\mu \, \delta \, \rho$. Also, let $\mu \in I^X$ be such that $\mu \, \delta \, \rho$ for each $\rho \in \sigma_x$. If $\rho \in I^X$ is such that $\mu \, \bar{\delta} \, \rho$, then $\rho \notin \sigma_x$. Thus $\bar{\rho}(x) \neq 1$, which implies (by Proposition (2.1)) that $\bar{\rho}(x) = 0$ and so $\rho(x) = 0$. Thus

$$\sup\{\rho(x) : \rho \, \bar{\delta} \, \mu\} = 0$$

and therefore $\bar{\mu}(x) = 1$, which shows that $\mu \in \sigma_x$.

Finally, suppose that $\mu \vee \rho \in \sigma_x$. Then

$$1 = \overline{\mu \vee \rho}(x) = \bar{\mu}(x) \vee \bar{\rho}(x)$$

and thus either $\mu \in \sigma_x$ or $\rho \in \sigma_x$.

5.3. Lemma. *Let (X, δ) be a fuzzy proximity space. Then*

(1) *If σ_1, σ_2 are δ-clusters on X with $\sigma_1 \subset \sigma_2$, then $\sigma_1 = \sigma_2$.*

(2) *Let σ be a δ-cluster. Then*

(a) *$\mu_1 \geqslant \mu \in \sigma$ implies $\mu_1 \in \sigma$.*

(b) *$0 \notin \sigma$ and $1 \in \sigma$.*

(c) *$\mu \in \sigma$ iff $\bar{\mu} \in \sigma$.*

(d) *If there exists a $\mu \in \sigma$ and an $x \in X$ such that $\mu(y) = 0$ if $y \neq x$, then $\sigma = \sigma_x$.*

Proof. (1) Let $\mu \in \sigma_2$. If $\rho \in \sigma_1$, then μ, ρ are both in σ_2 and thus $\mu \, \delta \, \rho$ by (C1). This, by (C2), implies that $\mu \in \sigma_1$.

(2) (a) Since $\mu \in \sigma$, we have $\mu \, \delta \, \rho$ for each $\rho \in \sigma$. This implies that $\mu_1 \, \delta \, \rho$ for each $\rho \in \sigma$ and hence $\mu_1 \in \sigma$.

(b) This follows from (C1) and (C2).

(c) This follows from (C2) since $\bar{\mu} \, \delta \, \rho$ implies $\mu \, \delta \, \rho$.

(d) Let μ and x be a such that $\mu(y) = 0$ if $y \neq x$.

Let $\rho \in \sigma$. Then $\mu\, \delta\, \rho$. Let ρ_1 be such that $\rho_1\, \bar{\delta}\, \rho$. Then $\bar{\rho}_1\, \bar{\delta}\, \rho$ and hence $\bar{\rho}_1(x) = 0$. (If $\bar{\rho}_1(x) \neq 0$, then $\bar{\rho}_1(x) = 1$ and hence $\bar{\rho}_1 \geqslant \mu$, which would would imply that $\mu\, \bar{\delta}\, \rho$), and thus $\rho_1(x) = 0$. Therefore

$$\sup\{\rho_1(x) \colon \rho_1\, \bar{\delta}\, \rho\} = 0,$$

which gives that $\bar{\rho}(x) = 1$ and so $\rho \in \sigma_x$. Thus $\sigma \subset \sigma_x$ and consequently $\sigma = \sigma_x$ by (1).

5.4. Lemma. *Let (X, δ) be a fuzzy proximity space and $\sigma \subset I^X$ such that*

(a) $0 \notin \sigma$.

(b) $\mu \in \sigma$ *iff* $\bar{\mu} \in \sigma$.

(c) $\mu \vee \rho \in \sigma$ *iff* $\mu \in \sigma$ *or* $\rho \in \sigma$.

Let $\mu_0 \in \sigma$. Then, there exists a fuzzy ultrafilter φ on X with $\mu_0 \in \varphi \subset \sigma$.

Proof. Let Ω denote the family of all subsets ω of I^X with the following two properties:

(1) $\mu_0 \in \omega$.

(2) If $\mu_1, \ldots, \mu_n \in \omega$, then $\mu_1 \wedge \cdots \wedge \mu_n \in \sigma$.

By Zorn's lemma, there is a maximal (with respect to set inclusion) member ω_0 of Ω. We will show that ω_0 is a fuzzy filter on X. Clearly

$$\mu_0 \in \omega_0 \subset \sigma.$$

Let $\mu_1, \mu_2 \in \omega_0$. Then $\mu_1 \wedge \mu_2 \in \sigma$. Since $\omega_0 \cup \{\mu_1 \wedge \mu_2\} \in \Omega$ we will have $\mu_1 \wedge \mu_2 \in \omega_0$ by the maximality of ω_0. Also, let $\mu_1 \geqslant \mu \in \omega_0$. Then $\mu_1 = \mu_1 \vee \mu \in \sigma$. Since $\omega_0 \cup \{\mu_1\} \in \Omega$, we have $\mu_1 \in \omega_0$. This proves that ω_0 is a fuzzy filter on X. Let φ be a fuzzy ultrafilter on X containing ω_0. We will finish the proof by showing that $\varphi \subset \sigma$. Suppose, by way of contradiction, that there exists a $\mu \in \varphi$ which is not in σ. By (b), $\bar{\mu} \notin \sigma$. Also, since $\bar{\mu} \in \varphi$ and $\bar{\mu} \wedge (1 - \bar{\mu}) = 0$, we have that $1 - \bar{\mu} \notin \varphi$ and so $1 - \bar{\mu} \notin \omega_0$. Since $1 = \bar{\mu} \vee (1 - \bar{\mu}) \in \sigma$ and $\bar{\mu} \notin \sigma$, we have $1 - \bar{\mu} \in \sigma$. Since $1 - \bar{\mu} \notin \omega_0$, we could not have $(1 - \bar{\mu}) \wedge \rho \in \sigma$ for all $\rho \in \omega_0$. In fact, suppose that $(1 - \bar{\mu}) \wedge \rho \in \sigma$ for each $\rho \in \omega_0$. If $\rho_1, \ldots, \rho_n \in \omega_0$, then $\rho_1 \wedge \cdots \wedge \rho_n \in \omega_0$ (since ω_0 is a fuzzy filter) and thus $\rho_1 \wedge \rho_2 \wedge \cdots \wedge \rho_n \wedge (1 - \bar{\mu}) \in \sigma$ by our hypothesis. It follows that $\omega_0 \cup \{1 - \bar{\mu}\} \in \Omega$ and so $1 - \bar{\mu} \in \omega_0$, which is a contradiction. Thus, there exists $\rho \in \omega_0$ with $\rho \wedge (1 - \bar{\mu}) \notin \sigma$. Also $\bar{\mu} \wedge \rho \notin \sigma$ since $\bar{\mu} \notin \sigma$ and $\bar{\mu} \wedge \rho \leqslant \bar{\mu}$. Therefore

$$\rho = 1 \wedge \rho = [\bar{\mu} \vee (1 - \bar{\mu})] \wedge \rho = (\bar{\mu} \wedge \rho) \vee [(1 - \bar{\mu}) \wedge \rho] \notin \sigma$$

by (c), which is a contradiction. This completes the proof.

5.5. THEOREM. *Let* (X, δ) *be a fuzzy proximity space and* $\sigma \subset I^X$. *Then:*

(1) σ *is a* δ*-cluster iff there exists a fuzzy ultrafilter* φ *on* X *such that*

$$\sigma = \{\mu \in I^X \colon \mu \ \delta \ \rho \text{ for every } \rho \in \varphi\}.$$

(2) *If* σ *is a* δ*-cluster and* $\mu_0 \in \sigma$, *then there exists a fuzzy ultrafilter* φ *on* X *with* $\mu_0 \in \varphi \subset \sigma$.

(3) *If* σ *is a* δ*-cluster and* φ *a fuzzy ultrafilter with* $\varphi \subset \sigma$, *then*

$$\sigma = \{\mu \colon \mu \ \delta \ \rho \text{ for each } p \in \varphi\}.$$

Proof. Since every δ-cluster σ on X has properties (a), (b), (c) of Lemma 5.4, (2) follows from Lemma 5.4. To prove (1), suppose that there exists a fuzzy ultrafilter φ on X such that

$$\sigma = \{\mu \colon \mu \ \delta \ \rho \text{ for each } \rho \in \varphi\}.$$

We will show that σ is a δ-cluster. In fact, let $\mu_1, \mu_2 \in \sigma$ and assume, by way of contradictions that $\mu_1 \ \bar{\delta} \ \mu_2$. Then, there exists ρ with $\mu_1 \ \bar{\delta} \ \rho$ and $(1 - \rho) \ \bar{\delta} \ \mu_2$. Since $\mu_1, \mu_2 \in \sigma$, we have that ρ, $1 - \rho \notin \varphi$. This is a contradiction by Proposition 4.3. This proves that σ satisfies (C1). For (C2), assume that $\mu \ \delta \ \rho$ for each $\rho \in \sigma$. If $\rho \in \varphi$, then $\rho \wedge \rho_1 \neq 0$ for each $\rho_1 \in \varphi$. Hence $\rho \ \delta \ \rho_1$ for each $\rho_1 \in \varphi$, which implies that $\rho \in \sigma$. Thus $\varphi \subset \sigma$. Since $\mu \ \delta \ \rho$ for each $\rho \in \sigma$ we will have $\mu \ \delta \ \rho$ for each $\rho \in \varphi$ and so $\mu \in \sigma$. Finally, let $\mu_1, \mu_2 \notin \sigma$. There are $\rho_1, \rho_2 \in \varphi$ with $\mu_1 \ \bar{\delta} \ \rho_1$ and $\mu_2 \ \bar{\delta} \ \rho_2$. If $\rho = \rho_1 \wedge \rho_2$, then $\rho \in \varphi$ and $(\mu_1 \vee \mu_2) \ \bar{\delta} \ \rho$ which implies that $\mu_1 \vee \mu_2 \notin \sigma$. Thus σ is a δ-cluster.

Conversely, assume that σ is a δ-cluster and let $\mu_0 \in \sigma$. Then, there exists a fuzzy ultrafilter φ on X with $\mu_0 \in \varphi \subset \sigma$. Let

$$\sigma_1 = \{\mu \colon \mu \ \delta \ \rho \text{ for each } p \in \varphi\}.$$

By what we proved, σ_1 is a δ-cluster. If $\mu \in \sigma$, then $\mu \ \delta \ \rho$ for each $\rho \in \sigma$. In particular, $\mu \ \delta \ \rho$ for each $\rho \in \varphi$ and so $\mu \in \sigma_1$. Thus $\sigma \subset \sigma_1$ and hence $\sigma = \sigma_1$ by Lemma 5.3.

Finally, assume that σ is a δ-cluster and φ a fuzzy ultrafilter such that $\varphi \subset \sigma$. If

$$\sigma_1 = \{\mu \colon \mu \ \delta \ \rho \text{ for each } \rho \in \varphi\}.$$

then σ_1 is a δ-cluster containing σ and hence $\sigma = \sigma_1$.

5.6. COROLLARY. *Let* (X, δ) *be a fuzzy proximity space. Then, every fuzzy ultrafilter* φ *on* X *is contained in a unique* δ*-cluster.*

Proof. If φ is a fuzzy ultrafilter on X, then φ is contained in the unique δ-cluster $\sigma = \{\mu \colon \mu \ \delta \ \rho \text{ for each } \rho \in \varphi\}$.

5.7. Proposition. *Let (X, δ) be a fuzzy proximity space and $\mu_0, \rho_0 \in I^X$. The following two assertions are equivalent:*

(1) $\mu_0 \delta \rho_0$.

(2) *There exists a δ-cluster σ which contains both μ_0 and ρ_0.*

Proof. $(2 \Rightarrow 1)$ This is obvious.
$(1 \Rightarrow 2)$ Let

$$\sigma_0 = \{\mu : \mu \delta \rho_0\}.$$

Then σ_0 has the following properties:

(a) $\mu_0 \in \sigma_0$.

(b) $0 \notin \sigma_0$.

(c) $\mu \in \sigma_0$ iff $\bar{\mu} \in \sigma_0$.

(d) $\mu \vee \rho \in \sigma_0$ iff $\mu \in \sigma_0$ or $\rho \in \sigma_0$.

By Lemma 5.4, there exists a fuzzy ultrafilter φ on X with $\mu_0 \in \varphi \subset \sigma_0$. Let

$$\sigma = \{\mu : \mu \delta \rho \text{ for every } \rho \in \varphi\}.$$

The δ-cluster σ contains both μ_0 and ρ_0. In fact, if $\rho \in \varphi$, then $\rho \in \sigma_0$ and so $\rho \delta \rho_0$. Thus $\rho_0 \delta \rho$ for each $\rho \in \varphi$ and hence $\rho_0 \in \sigma$. Also, since $\varphi \subset \sigma$, $\mu_0 \in \sigma$.

5.8. Theorem. *Let (X, δ_1) and (Y, δ_2) be fuzzy proximity spaces and $f: X \to Y$ a proximity map. If σ_1 is a δ_1-cluster on X, then $\sigma_2 = \{\mu \in I^Y : \mu \delta_2 f(\rho)$ for each $\rho \in \sigma_1\}$ is a δ_2-cluster on Y which contains $f(\sigma_1)$.*

Proof. By Theorem 5.5, there is a fuzzy ultrafilter φ on X such that

$$\sigma_1 = \{\mu \in I^X : \mu \delta_1 \rho \text{ for each } \rho \in \varphi\}.$$

By Proposition 4.4, $f(\varphi)$ is a base for a maximal ultrafilter φ' on Y. We will show that σ_2 coincides with the δ_2-cluster

$$\sigma_0 = \{\mu \in I^Y : \mu \delta_2 \rho \text{ for each } p \in \varphi'\}.$$

In fact, let $\mu \in \sigma_2$. Since $\varphi \subset \sigma_1$, we have $\mu \delta_2 f(\rho)$ for each $\rho \in \varphi$ and hence $\mu \in \sigma_0$. Thus $\sigma_2 \subset \sigma_0$. To prove the inverse inclusion, we first observe that $f(\sigma_1) \subset \sigma_0$ since f is a proximity map. Thus, if $\rho \in \sigma_0$, then for each $\mu \in \sigma_1$ we have ρ, $f(\mu) \in \sigma_0$ and so $\rho \delta_2 f(\mu)$. Thus $\rho \in \sigma_2$ and this completes the proof.

Let (X, δ) be a fuzzy poximity space and Y a nonempty subset of X. The fuzzy proximity δ_Y induced by δ on Y is the coarsest fuzzy proximity on Y for which the inclusion map from Y to X is proximally continuous (see [5], 5.1). If, for each $\mu \in I^Y$, we denote by $\tilde{\mu}$ the element of I^X which coincides with μ on Y and vanishes on $X - Y$, then ([5], 5.2), for $\mu, \rho \in I^Y$, we have $\mu \delta_Y \rho$ iff $\mu_X \delta \rho_X$. If

$f: Y \to X$ is the inclusion map, then $f(\mu) = \mu_X$. We thus have the following corollary to Theorem 5.7.

5.9. COROLLARY. *Let* (X, δ) *be a fuzzy proximity space and* $Y \subset X$. *If* σ_1 *is a* δ_Y*-cluster on* Y, *then* $\sigma_2 = \{\mu \in I^X: \mu\ \delta\ \rho_X$ *for each* $\rho \in \sigma_1\}$ *is a* δ*-cluster on* X.

5.10. THEOREM. *Let* σ *be a* δ*-cluster on a fuzzy proximity space* (X, δ), $Y \subset X$ *and suppose that there exists a* $\mu_0 \in \sigma$ *which vanishes on* $X - Y$. *Let* $\sigma_1 = \{\mu \in \sigma: \mu = 0$ *on* $X - Y\}$ *and* $\sigma' = \{\tilde{\mu}: \mu \in \sigma_1\}$, *where for* $\mu \in I^X$, $\tilde{\mu} = \mu \mid Y$. *Then* σ' *is a* δ_Y*-cluster on* Y.

Proof. By Theorem 5.5, there exists a fuzzy ultrafilter φ on X containing μ_0 and such that

$$\sigma = \{\mu \in I^X: \mu\ \delta\ \rho \text{ for each } \rho \in \varphi\}.$$

Let $\tilde{\varphi} = \{\tilde{\mu}: \mu \in \varphi\}$. By Proposition (4.5), $\tilde{\varphi}$ is a fuzzy ultrafilter on Y. Hence the set

$$\sigma_0 = \{\mu \in I^Y: \mu\ \delta_Y\ \rho \text{ for each } p \in \tilde{\varphi}\}$$

is a δ_Y-cluster on Y. We will show that $\sigma_0 = \sigma'$. In fact, let $\rho \in \sigma_0$. Then $\rho\ \delta_Y\ \tilde{\mu}$ for each $\mu \in \varphi$. Since $\mu \geqslant (\tilde{\mu})_X$, we have $\rho_X\ \delta\ \mu$ for each $\mu \in \varphi$ and thus $\rho_X \in \sigma_1$ since we also have $\rho_X = 0$ on $X - Y$. Therefore $\rho = \tilde{\rho}_X \in \sigma'$. This proves that $\sigma_0 \subset \sigma'$. For the inverse inclusion, let $\rho \in \sigma_1$. Let $\mu \in \varphi$. Then $\mu_1 = \mu \wedge \mu_0 \in \varphi$ and thus $\rho\ \delta\ \mu_1$. Since $(\tilde{\rho})_X = \rho$ and $(\tilde{\mu}_1)_X = \mu_1$, from the $\rho\ \delta\ \mu_1$ it follows that $\tilde{\rho}\ \delta_Y\ \tilde{\mu}_1$ and thus $\tilde{\rho}\ \delta_Y\ \tilde{\mu}$ since $\tilde{\mu} \geqslant \tilde{\mu}_1$. Thus, for each $\tilde{\mu} \in \tilde{\varphi}$, we have $\tilde{\rho}\ \delta_Y\ \tilde{\mu}$, which implies that $\tilde{\rho} \in \sigma_0$. This finishes the proof.

5.11. DEFINITION. A fuzzy semiultrafilter on a set X is a nonempty collection φ of nonzero members of I^X satisfying the following conditions:

(1) If $\mu \vee \rho \in \varphi$, then $\mu \in \varphi$ or $\rho \in \varphi$.

(2) $\mu \geqslant \rho \in \varphi$ implies $\mu \in \varphi$.

(3) If $\mu \wedge \rho \neq 0$ for each $\rho \in \varphi$, then $\mu \in \varphi$.

Clearly every fuzzy ultrafilter is a fuzzy semiultrafilter. Also, if (X, δ) is a fuzzy proximity space, then every δ-cluster is a fuzzy semiultrafilter.

5.12. THEOREM. *Let* X *be a set and* Φ *a family of fuzzy semiultrafilters on* X. *The following are equivalent:*

(i) *There is a fuzzy proximity* δ *on* X *such that* Φ *coincides with the family of all* δ*-clusters.*

(ii) Φ *has the following properties:*

(a) *Every fuzzy ultrafilter is contained in a member of* Φ.

(b) *Let* $\mu, \rho \in I^X$. *If for each* $\gamma \in I^X$ *there exists a* $\varphi \in \Phi$ *such that either* $\mu, \gamma \in \varphi$ *or* $\rho, 1 - \gamma \in \varphi$, *then there is a* $\varphi' \in \Phi$ *containing both* μ *and* ρ.

(c) *Let* $\varphi \in \Phi$ *and* $\mu \in I^X$. *If for each* $\rho \in \varphi$ *there exists* $\varphi' \in \Phi$ *with* $\mu, \rho \in \varphi'$, *then* $\mu \in \varphi$.

Proof. (i $\Rightarrow$ ii)

(a) This follows from Corollary 5.6.

(b) Suppose that there is no $\varphi \in \Phi$ which contains both μ and ρ. By Proposition 5.7, we have $\mu \bar{\delta} \rho$. By (FP4), there exists a $\gamma \in I^X$ with $\mu \bar{\delta} \gamma$ and $(1 - \gamma) \bar{\delta} \rho$. In view of Proposition 5.7, there is no φ in Φ which contains either both the μ and γ or both the $1 - \gamma$ and ρ.

(c) Suppose that $\mu \notin \varphi \in \Phi$. Then, there exists $\rho \in \varphi$ with $\mu \bar{\delta} \rho$. For this ρ there is no $\varphi' \in \Phi$ which contains both μ and ρ.

(ii $\Rightarrow$ i). We define a binary relation δ on I^X by $\mu \delta \rho$ iff there exists a $\varphi \in \Phi$ which contains both μ and ρ. We will show first that δ is a fuzzy proximity on X.

It is easy to see that δ satisfies (FN1), (FN2), and (FN3). For the (FN4), suppose that $\mu \bar{\delta} \rho$. Then, there is no $\varphi \in \Phi$ containing both μ and ρ. By part (b) of (ii), there is a $\gamma \in I^X$ such that there is no φ in Φ which contains μ, γ or $1 - \gamma, \rho$. Hence, there is γ such that $\mu \bar{\delta} \gamma$ and $(1 - \gamma) \bar{\delta} \rho$.

Finally, suppose that $\mu \wedge \rho \neq 0$. Then, there is a fuzzy ultrafilter φ on X which contains both μ and ρ. By (a), there exists $\sigma \in \Phi$ with $\varphi \subset \sigma$. Now $\mu, \rho \in \sigma$ and hence $\mu \delta \rho$. This proves that δ is a fuzzy proximity on X.

Next we show that each member σ of Φ is a δ-cluster. Clearly σ satisfies (C1) and (C3). For (C2), suppose that $\mu \delta \rho$ for each $\rho \in \sigma$. Then, for each $\rho \in \sigma$ there exists $\sigma' \in \Phi$ with $\mu, \rho \in \sigma'$. By (c), $\mu \in \sigma$, which proves that (C2) also holds and thus σ is a δ-cluster. To finish the proof, it only remains to show that every δ-cluster belongs to Φ. So, let σ be a δ-cluster. There exists a fuzzy ultrafilter φ on X such that

$$\sigma = \{\mu : \mu \delta \rho \text{ for each } \rho \in \varphi\}.$$

By (a), there is a $\sigma_0 \in \Phi$ with $\varphi \subset \sigma_0$. Since σ_0 is a δ-cluster, we have

$$\sigma_0 = \{\mu : \mu \delta \rho \text{ for each } \rho \in \varphi\} = \sigma.$$

This completes the proof.

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